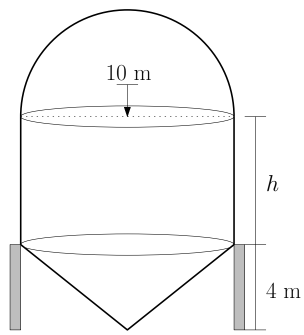
**Question 1**

Suppose you're building a silo that will be used to store sugar for use in an ice cream factory. The silo must hold 1000 cubic meters of sugar, and has dimensions as in the diagram below:




As you can see on the diagram, the silo is built from a cylinder, a cone, and half of a sphere. The height of the cylinder is unknown. In this exercise, you'll determine $h$, the height of the cylinder.

1. From the diagram, you can see the diameter of the cylinder is 10 m. What is the radius of the half-sphere that sits on top of the cylinder?

2. What are the volume formulas for a half-sphere, cylinder, and cone?

3. Calculate the volume of the half-sphere and cone.

4. The total volume is supposed to be 1000 cubic meters. Use this fact to write down an equation that will help you to determine $h$.

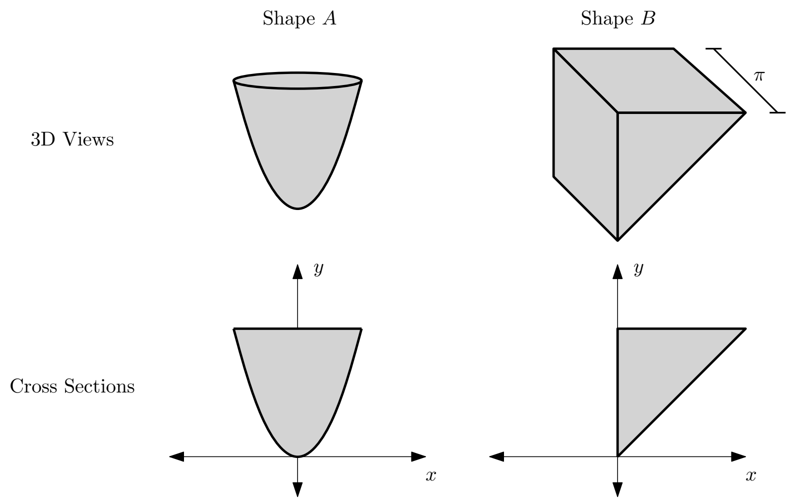
5. Solve your equation for $h$.

6. The engineers who will install the silo want to know the total height of the silo. Based on your calculations, what is the total height?

**Question 2**

In its most general form, *Cavalieri's principle* is a technique for arguing that shapes have equal volume (even if they *look* quite different). This exercise will walk you through the process of applying Cavalieri's principle to show a pair of shapes, $A$ and $B$, have equal volume.

Below is a picture of Shapes $A$ and $B$. You obtain Shape $A$ by taking the region contained between the parabola $y = x^2$ and the line $y = 4$, and rotating this shape around the $y$-axis to create a three dimensional "dome shaped" solid.




**Part 1:** What does a horizontal cross section of $A$ look like?

Next, we have a picture of $B$. You get this shape by taking the triangular region contained between the lines $y = x$, $x = 0$, and $y = 4$, and then "extruding it" to a thickness of $\pi$ to get a prism shaped solid.

**Part 2:** What does a horizontal cross section of the trough look like?

To apply Cavalieri's principle, we first notice that the two shapes are contained between the horizontal planes $P$ and $Q$ at heights $y = 0$ and $y = 4$. We consider the cross sections of the shapes when we slice them by another horizontal plane, at height $y = t$. The next step is to check that if we slice $A$ and $B$ with a plane parallel to $P$ and $Q$ (i.e. a horizontal plane), then the cross sections have equal area.

**Part 3:** The cross section of $A$ at height $y = t$ is a circle of some radius. What is the area of the circle in terms of $t$?

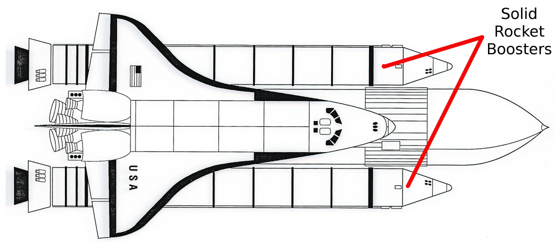
**Part 4:** The cross section of $B$ at height $t$ is a rectangle. What is the area of that rectangle in terms of $t$?

You should find that your answers to the last two questions are the same. Consequently, Cavalieri's principle tells you that shape $A$ and shape $B$ have equal volume.

**Part 5:** The volume of shape $A$ might be difficult to calculate on its own. The volume of shape $B$ is much easier to calculate. Determine the volume of shape $B$. Using your answer, what can you say about shape $A$?

**Question 3**

When NASA flew Space Shuttle missions, the shuttle was propelled into space by two *solid rocket boosters* and an *external tank* filled with liquid hydrogen and liquid oxygen.




In this exercise, you'll estimate the total volume of one of the solid rocket boosters. Model the booster as a cylinder topped with a circular cone. The cylinder part was about 4 meters in diameter and the total height of the booster (the distance from the base of the rocket booster to the tip of the nose cone) was about 45 meters. The height of the cone was about 8 meters.

1. Using the height of the cone, calculate the height of the cylindrical part of the booster.

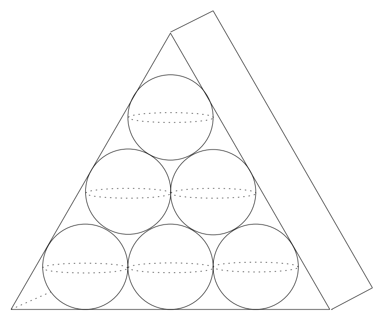
2. Determine the radius of the cylinder.

3. What are the formulas for the volumes of a right circular cone and a cylinder?

4. Using the information from the previous three parts, estimate the volume of the solid rocket booster.

**Question 4**

Francesca and Rodrigo like to play *bocce*, a ball tossing game, at the beach. Rodrigo has a set of 6 bocce balls, each 100 mm in diameter. The bocce balls come in a box shaped like the one shown below, with a lid. The triangle is equilateral and the box is 100 mm thick. As you can see, the bocce balls just barely fit into the box.



Francesca decides to pull a prank on Rodrigo. When he isn't looking, she fills the remaining space in the box with sand until the box is completely full, and replaces the lid. What is the total volume of the sand Francesca packed around the bocce balls?

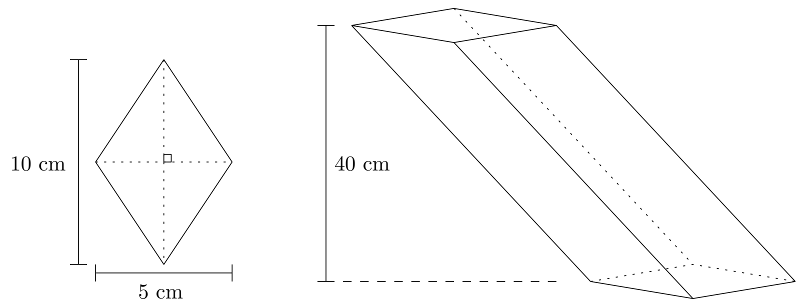
**Question 5**

The McMath-Pierce Solar Telescope is a special telescope located at Kitt Peak National Observator in Arizona. Unlike most telescope, this instrument is used just for studying the sun. Here is one view of the telescope:

Here is another:

As you can see, the telescope has a slanted portion, which is where the rays of light travel.

You've decided to build a diorama of the telescope and are working on building the slanted part of the telescope — we'll call this shape $S$. Below are your schematics:




Shape $S$ is about 40 cm tall (from the tabletop to the to the top of $S$). You notice that $S$ has the property that any horizontal slice of the shape looks like the diamond shown in the picture.

For artistic reasons, you're planning to build shape $S$ out of clay. Consequently, you'd like to know what volume of clay you're going to need to build $S$. Use Cavalieri's principle to find the volume of $S$.

**Question 1** This exercise is supposed to give students experience with decomposing a complicated shape into simpler shapes. It is identical to the problem treated in the film, so it may be a good idea to work on this exercise *before* showing the film, so that students have a chance to familiarize themselves with the problem.

**Part 1:** The half-sphere and cylinder meet at a circle of diameter 10 m, so the radius of the half-sphere is 5 m.

**Parts 2-3:**

$$\begin{aligned} V_{\text{half-sphere}} &= \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = \frac{2}{3}\pi 5^3 = \frac{250}{3}\pi \\ V_{\text{cylinder}} &= \pi r^2 h_{\text{cylinder}} \\ V_{\text{cone}} &= \frac{1}{3}\pi r^2 h_{\text{cylinder}} = \frac{1}{3}\pi 5^2 4 = \frac{100}{3}\pi \end{aligned}$$

**Parts 4-5:**

$$\begin{aligned} 1000 &= \frac{250}{3}\pi + \pi 25h + \frac{100}{3}\pi \\ \frac{1000}{25\pi} &= \frac{10}{3} + h + \frac{4}{3} \\ h &= \frac{1000}{25\pi} - \frac{4}{3} - \frac{10}{3} \\ &= \frac{1000}{25\pi} - \frac{4}{3} - \frac{10}{3} \\ &\approx 8.06 \end{aligned}$$

**Part 6:** The total height of the silo is given by:

$$5 + h + 4 \approx 17.06$$

The silo is about 17 meters tall.

**Question 2**

This problem will be difficult if students haven't seen Cavalieri's principle used to show volumes are equal prior to watching the film.

**Part 1-2:** Horizontal cross sections of $A$ look like circles. Horizontal cross sections of $B$ look like rectangles.

**Part 3:** For this part, it can be helpful to identify the radius on the "cross sections" graph. The radius is the $x$-coordinate of the graph when $t = y = x^2$. Hence $x = \sqrt{t}$. So the radius at height $t$ is $\sqrt{t}$. The area of the circle at that height is $\pi r^2 = \pi(\sqrt{t})^2 = \pi t$.

**Part 4:** At height $t$, the rectangular cross section has width $x = y = t$ and height $\pi$. Hence its area is $\pi t$.

**Part 5:** The volume of $B$ is the area of the triangle times the depth of the rough. The triangle is a 45-45-90 triangle with side length 2, so its area is $\frac{1}{2} \cdot 2 \cdot 2 = 2$. Hence the volume of $B$ is $2\pi$. Since $A$ and $B$ have the same volume, we now know $A$ has volume $2\pi$ too.

**Question 3**

In this problem, students will need to draw their own version of the rocket booster and label their drawing with measurments. The picture gives a rough idea of what the booster looks like, but this isn't quite the same thing as the mathematical model they need to analyze. In particular, it might be worth mentioning that the booster has a nozzel (or frustum) at the base that we aren't going to model. It might be worthwhile talking about why we would want to estimate the volume of the boosters, namely, they hold solid rocket fuel and the volume gives an idea of how much fuel was needed to propel the shuttle to space. Finally, the parts in this problem give a fair amount of "scaffolding" for students. For more capable students, one might consider posing the problem with only part 4.

**Part 1:** The height of the cylindrical part of the booster is the total height of the booster minus the height of the cone, that is $45 - 8 = 37$ meters.

**Part 2:** Since the diameter of the cylinder is 4 meters, the radius is 2 meters.

**Part 3:** We have:

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h_{\text{cone}} = \frac{1}{3}\pi 2^2 \frac{8}{=} \frac{32}{3}\pi$$
$$V_{\text{cylinder}} = \pi r^2 h_{\text{cylinder}} = \pi \cdot 2^2 \cdot 37 = \pi 148$$

**Part 4:** To get the total volume, we add the volumes of the pieces:

$$V_{\text{cone}} + V_{\text{cylinder}} = \pi \cdot (148 + 32/3) \approx 498$$

So the total volume of the rocket booster is about 498 cubic meters.

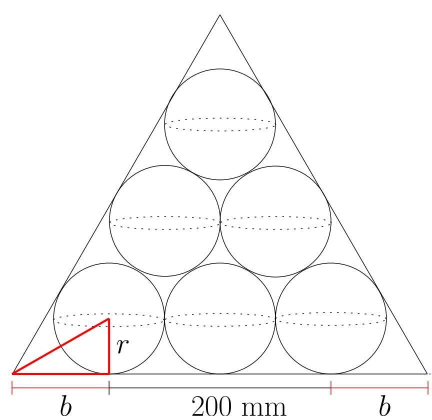
**Question 4**

This problem is a little challenging, for several reasons. First, it tests knowledge of equilateral triangles and 30-60-90 triangles. Second, the side length $\ell$ of the box is not known — students will need to solve for it. Third, students need to realize that they can subtract volumes, as well as adding them.

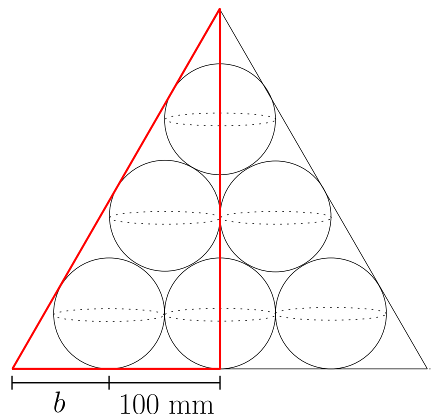
A good way to start is to recognize that the total volume of the box ($V_{\text{box}}$is equal to the volume of the sand, $V_{\text{sand}}$, plus the volume of the balls $V_{\text{balls}}$ — that's what it means for the box to be completely full! Notice that the radius of each ball is $r = 50$ mm, and use the formula for the volume of a sphere to write:

$$V_{\text{balls}} = 6 \cdot \frac{4}{3} \cdot \pi \cdot r^3 = 8\pi \cdot 50^3$$

Next we need to determine $V_{\text{box}}$, which means finding the area of the equilateral triangle defining the box. We can start by recognizing a small 30-60-90 triangle in the corner of the box:




The height of the triangle is $r = 50$. By the identities for these triangles, $b = \sqrt{3} \cdot 50$. Next we can recognize a bigger 30-60-90 triangle:




so that the equilateral triangle has base $200+2b$ and height $\sqrt{3}\cdot(100+b)$. Thus the area $A$ of the triangle is:

$$\begin{aligned} A &= \frac{1}{2}(200+2b)(\sqrt{3}(100+b)) \\ &= \sqrt{3}(100+50\sqrt{3})^2 \end{aligned}$$

The thickness of the box is 100 mm, so we have:

$$\begin{aligned} V_{\text{box}} &= 100A \\ &= 100\sqrt{3}(100+50\sqrt{3})^2 \end{aligned}$$

Thus, the volume of sand is:

$$\begin{aligned} V_{\text{sand}} &= V_{\text{box}} - V_{\text{balls}} \\ &= 100\sqrt{3}(100+50\sqrt{3})^2 - 8\pi \cdot 50^3 \approx 2.9 \times 10^6 \end{aligned}$$

Francesca put about $2.9 \times 10^6$ mm$^3$ (or about 2.9 Liters) of sand into the box.

**Question 5:**

This problem is a straightforward application of Cavalieri's principle. The volume of the model is 40 times the area of the diamond-shaped cross section. Students might have a little trouble seeing that they can calculate this area by multiplying the height and width dimensions. If this happens, you might consider cutting the diamond into four triangles and rearranging them to form a rectangle.

The final volume is $40 \cdot 5 \cdot 10 = 2000$ cm$^3$, or 2 litres of clay.